arXiv:math/0406563v2 [math.PR] 29 Jun 2004

Harnesses, Lévy bridges and *Monsieur Jourdain*

MANSUY Roger[*], YOR Marc

Laboratoire de Probabilités et modèles aléatoires,
Université Pierre et Marie Curie-Paris VI,
Casier 188,
4,place Jussieu,
F-75252 Paris Cedex 05

Abstract

Relations between harnesses ([Ham67], [Wil80]) and initial enlargements of the filtration of a Lévy process with its positions at fixed times are investigated.

Keywords : *Harnesses, Lévy processes, past-future martingales, enlargement of filtration*
AMS 2000 subject classification :*60G48, 60G51, 60G44, 60G10

Contents

1 Introduction

In order to model long-range misorientation within crystalline structure of metals, Hammersley [Ham67] introduced various notions of processes which enjoy particular conditional expectation properties. Among these, harnesses will be of particular interest. Let us precise the definition :

[*]contact : **mansuy@ccr.jussieu.fr**

Definition 1 *:*
Let $(H_t; t \geq 0)$ be a measurable process such that for all t, $\mathbb{E}[|H_t|] < \infty$, and define for all $t < T$:

$$\mathcal{H}_{t,T} := \sigma \{H_s; s \leq t; H_u; u \geq T\}$$

H is said to be a harness if, for all $a < b < c < d$

$$\mathbb{E}\left[\frac{H_c - H_b}{c - b} | \mathcal{H}_{a,d}\right] = \frac{H_d - H_a}{d - a} \tag{1}$$

One might also define the notion of $(\mathcal{F}_{t,T})_{t<T}$-harness as soon as $\mathcal{H}_{t,T} \subset \mathcal{F}_{t,T}$, with obvious hypothesis on a "past-future" filtration $\mathcal{F}$, which may be just as useful as the notion of Brownian motion with respect to a filtration. The equality may be reformulated as follows : H is a harness if and only if for all $s < t < u$

$$\mathbb{E}[H_t | \mathcal{H}_{s,u}] = \frac{t - s}{u - s} H_u + \frac{u - t}{u - s} H_s \tag{2}$$

Such a formulation justifies that Harnesses are sometimes called affine processes (See [CY03] chapter 6).
We note that Williams ([Wil73] and [Wil80]) proved the following striking result :the only squared integrable continuous harnesses are Brownian motions with drifts. This latter result shows how rigid the property of being a continuous harness is and may help understand why studies of harnesses with continuous time were so few during the past twenty years. On the other hand, some multi-parameter versions appeared, imitating Williams arguments (See [Doz81], [Zho92], [Zhu88] and [ZZ88]).
Glancing through the literature, it seems that no study of discontinuous harnesses has been performed. Our reference to Monsieur Jourdain (a character of Molière (1622-1673) [Poq70]) in the title alludes to this point; as Monsieur Jourdain discovers he was practising prose without being aware of it, the following theorem shows that a number of authors have been dealing with harnesses.

Theorem 2 *:*

(i) (Jacod-Protter, [JP88])Let $(\xi_t; t \geq 0)$ be an integrable Lévy process (that is: $\forall t, \mathbb{E}[|\xi_t|] < \infty$) and define

$$\mathcal{F}_{t,T} = \sigma \{\xi_s; s \leq t; \xi_u; u \geq T\}$$

Then for any given $T > 0$, there is the decomposition formula :

$$\xi_t = M_t^{(T)} + \int_0^t ds \, \frac{\xi_T - \xi_s}{T - s} \tag{3}$$

where $(M_t; t \leq T)$ is a $(\mathcal{F}_{t,T}; t \leq T)$-martingale

(ii) In a general framework, an integrable process $(H_t; t \geq 0)$ is a $(\mathcal{F}_{t,T})$-harness if and only if, for every $T > 0$, there exists $(M_t^{(T)})_{t<T}$ a $(\mathcal{F}_{t,T}; t < T)$-martingale such that

$$\forall t < T,\ H_t \ = \ M_t^{(T)} + \int_0^t ds\, \frac{H_T - H_s}{T - s} \tag{4}$$

For further results along this line, see Exercise 6.19 in [CY03] which provides a few references about harnesses. In the particular case of a Brownian motion ξ, formula (3) may be attributed to Itô [Itô78] but was already sketched by Lévy [Lév44a] and [Lév44b]. See also Jeulin-Yor [JY79]. Our motivation for writing this note is that harnesses -through formula (3)- seem to become more topical; indeed some recent works ([DNMBOP04] and [KHa04]) develop financial models of markets with well informed agents (also called insiders) where formula (4) plays a key-role. Some other papers ([FFNV] or [FN]) also deal with some notions of harness derived directly from the pioneering work of Hammersley, but are apparently far from the preceding discussion.
This note is organized as follows :

- First we prove part (ii) of the theorem.
- Section 3 is devoted to an alternative proof of the decomposition formula (3) of Jacod-Protter [JP88] thanks to the absolute continuity of the law of a Lévy process and its bridge.
- In Section 4, we develop the more general notion of past-future martingale and provide as many examples as possible.

2 Relations between Lévy bridges and harnesses

(2.1) Let $(B_t; t \geq 0)$ be a 1-dimensional Brownian motion; it is well known that a realization of the Brownian bridge over the time interval $[0, T]$, starting at x and ending at y, is:

$$\left\{x + \left(B_t - \tfrac{t}{T} B_T\right) + \tfrac{t}{T} y; t \leq T\right\} \tag{5}$$

Moreover, the semimartingale decomposition of this bridge is also well-known; it is the solution of the SDE :

$$X_t \ = \ x + \beta_t + \int_0^t ds\, \frac{y - X_s}{T - s}; \quad t \leq T \tag{6}$$

where $(\beta_t; t \leq T)$ is a standard Brownian motion.
This decomposition formula (6) is, in fact, equivalent to the semimartingale decomposition of $(B_t; t \leq T)$ in the enlarged filtration $\mathcal{B}_t^{(T)} := \mathcal{B}_t \vee \sigma(B_T)$, where $\mathcal{B}_t = \sigma\{B_s; s \leq t\}$:

$$B_t \ = \ \gamma_t^{(T)} + \int_0^t ds\, \frac{B_T - B_s}{T - s} \tag{7}$$

where $(\gamma_t^{(T)}; t \le T)$is a $(\mathcal{B}_t^{(T)}; t \le T)$-Brownian motion; in particular, it is independent of B_T. See [Itô78] and [JY79] for a discussion of (6) and (7).

(2.2) It has been shown by Jacod-Protter [JP88] that formula (7) in fact extends to any integrable Lévy process $(\xi_t; t \ge 0)$ in the following way :

$$\xi_t \quad = \quad M_t^{(T)} + \int_0^t ds\, \frac{\xi_T - \xi_s}{T - s} \tag{8}$$

where $(M_t^{(T)}; t \le T)$ is a martingale in the enlarged filtration $\mathcal{F}_t^{(T)} = \mathcal{F}_t \vee \sigma(\xi_T)$, where $\mathcal{F}_t = \sigma(\xi_s; s \le t)$.

(2.3) Here is the proof of part (ii) of Theorem 2:

a. $(\Rightarrow)$ Let H be a harness and $s < t < T$.

Define $M_t^{(T)} = H_t - \int\limits_0^t \frac{H_T - H_u}{T-u} du$.

Then, the harness property implies

$$\begin{aligned}
\mathbb{E}\left[M_t^{(T)} | \mathcal{F}_{s,T}\right] &= \mathbb{E}\left[H_t | \mathcal{F}_{s,T}\right] - \int\limits_0^s \frac{H_T - H_u}{T-u} du - \int\limits_s^t \mathbb{E}\left[\frac{H_T - H_u}{T-u} | \mathcal{F}_{s,T}\right] du \\
&= \frac{T-t}{T-s} H_s + \frac{t-s}{T-s} H_T - \int\limits_0^s \frac{H_T - H_u}{T-u} du - \int\limits_s^t \frac{H_T - H_s}{T-s} du \\
&= M_s^{(T)}
\end{aligned}$$

b. $(\Leftarrow)$ First, remark it is enough to show that, for all $s < t < T$

$$\mathbb{E}\left[\frac{H_t - H_s}{t-s} | \mathcal{F}_{s,T}\right] \quad = \quad \frac{H_T - H_s}{T-s} \tag{9}$$

Indeed, if $r < s < t < T$, then

$$\begin{aligned}
\mathbb{E}\left[\frac{H_t - H_s}{t-s} | \mathcal{F}_{r,T}\right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{H_t - H_s}{t-s} | \mathcal{F}_{s,T}\right] | \mathcal{F}_{r,T}\right] \\
&= \mathbb{E}\left[\frac{H_T - H_s}{T-s} | \mathcal{F}_{r,T}\right] \\
&= \frac{\mathbb{E}\left[H_r - H_s | \mathcal{F}_{r,T}\right]}{T-s} + \frac{H_T - H_r}{T-s} \\
&= \frac{r-s}{T-s} \frac{H_T - H_r}{T-r} + \frac{H_T - H_r}{T-s} \\
&= \frac{H_T - H_r}{T-r}
\end{aligned}$$

It only remains to prove formula (9). The assumed decomposition formula (4) yields to

$$H_t - H_s = M_t^{(T)} - M_s^{(T)} + \int_s^t dv \frac{H_T - H_v}{T - v}$$

Therefore

$$\begin{aligned}\mathbb{E}\left[H_t - H_s | \mathcal{F}_{s,T}\right] &= \int_s^t dv \frac{\mathbb{E}\left[H_T - H_v | \mathcal{F}_{s,T}\right]}{T - v} \\ &= \int_s^t \frac{dv}{T - v}\left(H_T - H_s\right) - \int_s^t \frac{dv}{T - v} \mathbb{E}\left[H_v - H_s | \mathcal{F}_{s,T}\right]\end{aligned}$$

Hence, s and T being fixed, $\phi(t) := \mathbb{E}\left[H_t - H_s | \mathcal{H}_{s,T}\right]$ solves the following first order linear differential equation :

$$\phi(t) = \int_s^t \frac{dv}{T - v}\left(H_T - H_s\right) - \int_s^t \frac{dv}{T - v}\phi(v);\ s \leq t \leq T$$

But this equation admits only one solution vanishing at s and a standard computation yields to $\phi(t) = \frac{H_T - H_s}{T - s}(t - s)$ which is formula (9).

Remark 3 *:*
Contrary to the very definition of harness, this proposition exhibits a privileged direction of time. So a similar representation property with the opposite time-direction can be derived. Namely, a measurable process H is a harness on $[0, T]$, if and only if, for all $T > \tau > 0$, there exists $(N_t^{(\tau)}; \tau < t \leq T)$ a $(\mathcal{F}_{\tau,t}; \tau < t \leq T)$-reverse martingale such that

$$\forall \tau < t \leq T,\ H_t = N_t^{(\tau)} - \int_t^T ds \frac{H_\tau - H_s}{\tau - s} \tag{10}$$

3 A Girsanov proof of the decomposition formula

(3.1) It is well known (see e.g. [FPY93]) that the law of the bridge of a Markov process is locally equivalent to the law of the "good" Markov process, more precisely, if X is a Markov process with $p_t(x, y)$ as its semigroup density from x to y, then the following absolute continuity relationship between $\mathbb{P}^t_{x \to y}$, the law of the bridge of length t from x to y and $\mathbb{P}_x$ the law of X starting at x holds :

$$\mathbb{P}^t_{x \to y | \mathcal{F}_s} = \frac{p_{t-s}(X_s, y)}{p_t(x, y)} . \mathbb{P}_{x | \mathcal{F}_s} \tag{11}$$

If ξ is a Lévy process, $\phi_t(.)$ will denote the density of the law of ξ_t, assuming it exists (see [Sat99] for conditions on a Lévy process to have such a density). The equality (11) then becomes :

$$\mathbb{P}^t_{x \to y|\mathcal{F}_s} = \frac{\phi_{t-s}(y-\xi_s)}{\phi_t(y-x)}.\mathbb{P}_{x|\mathcal{F}_s} \tag{12}$$

We now stay in the context of a Lévy process.

Lemma 4 *:*
If $(M^y_t; t \le T, y \in \mathbb{R})$ denote a family of variables such that

- *for any $y \in \mathbb{R}$, $(M^y_t; t \le T)$ is a $P^T_{x\to y}$-martingale.*
- *$(t,y) \mapsto M^y_t$ is measurable.*

Then $(M^{\xi_T}_t; t \le T)$ remains a P_x-martingale with respect to the filtration initially enlarged with ξ_T.

Proof :
Let $(M^y_t; t \le T, y \in \mathbb{R})$ be such a family of $P^T_{x\to y}$-martingales; then, for all $s < t < T$ and $\Gamma_s \in \sigma(\xi_u; u \le s)$,

$$\mathbb{E}^T_{x\to y}\left[1_{\Gamma_s}(M^y_t - M^y_s)\right] = 0$$

This implies, for any bounded Borel function f,

$$\int \mathbb{P}_x(\xi_T \in dy) f(y) \mathbb{E}^T_{x\to y}\left[1_{\Gamma_s}(M^y_t - M^y_s)\right] = 0$$

Therefore

$$\mathbb{E}_x\left[f(\xi_T) 1_{\Gamma_s}(M^{\xi_T}_t - M^{\xi_T}_s)\right] = 0$$

So, $M^{\xi_T}_t$ a P_x-martingale with respect to the filtration enlarged with ξ_T.

■

(3.2) If we suppose, without any loss of generality, that $\mathbb{E}[\xi_1] = 0$, then ξ is a P_x-martingale (in any other case, we will study the Lévy process $\xi_t - \mathtt{d}t$ where $\mathtt{d}$ is the drift term of ξ). We shall denote (σ^2, ν) its local characteristics (Brownian term and Lévy measure) and $\mathcal{L}$ its infinitesimal generator. For the sake of simplicity, note that $\tilde{\mathcal{L}}$, the infinitesimal generator of the time-space process (t, ξ_t) satisfies

$$\tilde{\mathcal{L}} = \frac{\partial}{\partial t} + \mathcal{L}$$

Thanks to the Girsanov theorem and the absolute continuity relationship (12), the process

$$\xi_t - \int_0^t \frac{d\langle \xi_\cdot, \phi_{T-\cdot}(y-\xi_\cdot)\rangle_s}{\phi_{T-s}(y-\xi_s)}$$

defines a $P^T_{x\to y}$-martingale and therefore

$$\xi_t - \int_0^t \frac{d\langle \xi_\cdot, \phi_{T-\cdot}(\xi_T-\xi_\cdot)\rangle_s}{\phi_{T-s}(\xi_T-\xi_s)}$$

is a P_x-martingale with respect to the filtration enlarged with ξ_T; this process will now be compared with $(M_t^{(T)})_{t\leq T}$ in part (ii) of Theorem 2. Namely, we aim to prove that

$$\langle \xi_\cdot, \phi_{T-\cdot}(y-\xi_\cdot)\rangle_t = \int_0^t \frac{y-\xi_s}{T-s}\, \phi_{T-s}(y-\xi_s)ds \tag{13}$$

that is, with our notation :

$$\tilde{\mathcal{L}}(x\phi_{T-s}(y-x))(s,x) = \frac{y-x}{T-s}\phi_{T-s}(y-x)$$

Now,

$$\tilde{\mathcal{L}}(x\phi_{T-s}(y-x))(s,x) = -\sigma^2\phi'_{T-s}(y-x) + \int \nu(dz)z\phi_{T-s}(y-x-z)$$

[This computation is quite easy once we note that $(t,x)\mapsto \phi_{T-t}(y-x)$ is a space-time harmonic function.]
The following lemma concludes the proof :

Lemma 5 *:*
For any integrable Lévy process with local characteristics (σ^2,ν) and transition probability density ϕ,

$$-\sigma^2\phi'_u(x) + \int \nu(dz)z\phi_u(x-z) = \frac{x}{u}\phi_u(x) \tag{14}$$

Proof :
From the very definition of the Lévy exponent, we have :

$$\int e^{i\lambda x}\phi_u(x)dx = \mathbb{E}\left[e^{i\lambda\xi_u}\right] = e^{-u\Phi(\lambda)} \tag{15}$$

Differentiation in λ within this equality yields to

$$i\int x\phi_u(x)e^{i\lambda x}dx = -u\Phi'(\lambda)e^{-u\Phi(\lambda)}$$

with $\Phi'(\lambda) = \sigma^2\lambda - i\int \nu(dz) z e^{i\lambda z}$
Replacing $e^{-u\Phi(\lambda)}$ with the expression in (15) and noting that

$$\begin{aligned}
\lambda \int \phi_u(x) e^{i\lambda x} dx &= i \int \phi'_u(x) e^{i\lambda x} dx \\
\int \nu(dz) z e^{i\lambda z} \int \phi_u(x) e^{i\lambda x} dx &= \int dx e^{i\lambda x} \int \nu(dz) z \phi_u(x-z)
\end{aligned}$$

we obtain :

$$i \int x\phi_u(x) e^{i\lambda x} dx = -u \int dx \left(-\sigma^2 \phi'_u(x) + \int \nu(dz) z \phi_u(x-z) \right) e^{i\lambda x}$$

■

Remark 6 *:*
The right-hand side of (14) can also be interpreted, for skip-free Lévy processes, as the density of the first hitting time thanks to Kendall's identity (See e.g. [BB01]).

4 A wider class of processes: the past-future martingales

(4.1) If $\mathcal{F}$ denotes a past-future filtration, the following definition generalizes the notion of a $\mathcal{F}$-harness :

Definition 7 *:*
The two-parameters process $(M_{s,t})_{0\leq s<t<\infty}$ is said to be a past-future martingale with respect to $(\mathcal{F}_{s,t})_{0\leq s<t<\infty}$ if :

1. $\forall s < t$, $\mathbb{E}[|M_{s,t}|] < \infty$
2. $\forall s < t$, $M_{s,t}$ *is* $\mathcal{F}_{s,t}$*-measurable.*
3. $\forall r < s < t < u$, $\mathbb{E}[M_{s,t} | \mathcal{F}_{r,u}] = M_{r,u}$

Remark 8 *:*

- *As previously mentioned, a process H is a $\mathcal{F}$-harness if and only if $\left(\frac{H_t - H_s}{t-s}\right)_{0\leq s<t<\infty}$ is a past-future martingale.*
- *Note that past-future martingales are reverse martingales indexed by the intervals of $\mathbb{R}^+$.*

(4.2) Here we are to detail some non trivial past-future martingales related to a standard Brownian motion $(B_t; t \geq 0)$.

1. Let f_+ and f_- be two both square-integrable and integrable functions on $\mathbb{R}^+$ and $C \in \mathbb{R}$. Then the process $(M_{s,t})_{0 \leq s < t < \infty}$ defined for all $s < t$ by :
$$\begin{aligned} M_{s,t} &= \int_0^s f_-(u)dB_u + \int_t^\infty f_+(u)dB_u + \dots \\ &\dots + \frac{B_t - B_s}{t-s}\left(C - \int_0^s f_-(u)du - \int_t^\infty f_+(u)du\right) \end{aligned}$$
is a past-future Brownian martingale.
One notices that the stochastic integral terms associated to the functions $f_\pm$ have to be "compensated" with a harness term.

2. An exponential example can easily be derived from this latter. Within the same framework, the two-parameter process $(N_{s,t})_{0 \leq s < t\infty}$ defined for all $s < t$
$$\begin{aligned} \ln N_{s,t} &= M_{s,t} + \frac{1}{2}\int_0^s f_-^2(u)du + \frac{1}{2}\int_t^\infty f_+^2(u)du + \dots \\ &\dots + \frac{t-s}{2}\left(C - \int_0^s f_-(u)du - \int_t^\infty f_+(u)du\right)^2 \end{aligned}$$
is a past-future martingale.

(4.3) Previous examples can easily be extended to more general Lévy processes :

Proposition 9 *:*
Let ξ be a Lévy process and f an integrable function with locally finite variation, chosen to be right-continuous with left limits, such that $\int_0^\infty f(u-)d\xi_u$ exists. Then, for all $s < t$,
$$M_{s,t} = \int_0^s f(u^-)d\xi_u + \int_t^\infty f(u^-)d\xi_u + \frac{\xi_t - \xi_s}{t-s}\int_s^t f(u)du + [\xi_\centerdot, f(\centerdot)]_s - [\xi_\centerdot, f(\centerdot)]_t$$
defines a past-future martingale.

Proof :
Indeed thanks to integration by parts formula
$$\begin{aligned} \mathbb{E}\left[\int_s^t f(u^-)d\xi_u | \xi_t, \xi_s\right] &= f(t^-)\xi_t - f(s)\xi_s - \int_s^t \mathbb{E}\left[\xi_{u^-}|\xi_t, \xi_s\right] df(u) + [\xi_\centerdot, f(\centerdot)]_s - [\xi_\centerdot, f(\centerdot)]_t \\ &= \xi_t\left(f(t^-) - \int_s^t \frac{u-s}{t-s}df(u)\right) + \xi_s\left(-f(s) - \int_s^t \frac{t-u}{t-s}df(u)\right) \\ &\quad \dots + [\xi_\centerdot, f(\centerdot)]_s - [\xi_\centerdot, f(\centerdot)]_t \\ &= \frac{(\xi_t - \xi_s)}{t-s}\int_s^t f(u)du + [\xi_\centerdot, f(\centerdot)]_s - [\xi_\centerdot, f(\centerdot)]_t \end{aligned}$$

Therefore

$$M_{s,t} = \mathbb{E}\left[\int_0^\infty f(u^-)d\xi_u|\mathcal{H}_{s,t}\right]$$

■

References

[BB01] Konstantin Borovkov and Z. Burq. Kendall's identity for the first crossing time revisited. *Electron. Comm. Probab.*, 6:91–94 (electronic), 2001.

[CY03] Loïc Chaumont and Marc Yor. *Exercises in probability, a guided tour from measure theory to random processes, via conditioning.* Cambridge series in statistical and probabilistic mathematics. Cambridge University Press, 2003.

[DNMBOP04] Giulia Di Nunno, Thilo Meyer-Brandis, Bernt Øksendal, and Frank Proske. Optimal portfolio for an insider in a market driven by Lévy processes. *Submitted to SPA*, 2004.

[Doz81] Marco Dozzi. Two-parameter harnesses and the Wiener process. *Z. Wahrsch. Verw. Gebiete*, 56(4):507–514, 1981.

[EY04] Michel Emery and Marc Yor. A parallel between brownian bridges and gamma bridges. Technical Report 846, Laboratoire de probabilités et modèles aléatoires, September 2003, To appear in Publ. RIMS, Kyoto(2004).

[FFNV] Pablo A. Ferrari, Luiz R. G. Fontes, Beat. M. Niederhauser, and Marina Vachkovskaia. The serial harness interacting with a wall. arXiv:math.PR/0210218.

[FN] Pablo A. Ferrari and Beat M. Niederhauser. Harness processes and harmonic crystals. arXiv:math.PR/0312402.

[FPY93] Pat Fitzsimmons, Jim Pitman, and Marc Yor. Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 101–134. Birkhäuser Boston, Boston, MA, 1993.

[Ham67] John M. Hammersley. Harnesses. In *Proc. Fifth Berkeley Sympos. Mathematical Statistics and Probability (Berkeley, Calif., 1965/66), Vol. III: Physical Sciences*, pages 89–117. Univ. California Press, Berkeley, Calif., 1967.

[HM94] John M. Hammersley and G. Mazzarino. Properties of large Eden clusters in the plane. *Combin. Probab. Comput.*, 3(4):471–505, 1994.

[Itô78] Kiyosi Itô. Extension of stochastic integrals. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 95–109, New York, 1978. Wiley.

[JP88] Jean Jacod and Philip Protter. Time reversal on Lévy processes. *Ann. Probab.*, 16(2):620–641, 1988.

[JY79] Thierry Jeulin and Marc Yor. Inégalité de Hardy, semimartingales, et faux-amis. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Math.*, pages 332–359. Springer, Berlin, 1979.

[KHa04] Arturo Kohatsu-Higa and al. In preparation. Technical report, 2004.

[Kin86] John F. C. Kingman. The construction of infinite collections of random variables with linear regressions. *Adv. in Appl. Probab.*, (suppl.):73–85, 1986.

[Lév44a] Paul Lévy. Un théorème d'invariance projective relatif au mouvement brownien. *Comment. Math. Helv.*, 16:242–248, 1944.

[Lév44b] Paul Lévy. Une propriété d'invariance projective dans le mouvement brownien. *C. R. Acad. Sci. Paris*, 219:378–379, 1944.

[Mey94] Paul-André Meyer. Sur une transformation du mouvement brownien due à Jeulin et Yor. In *Séminaire de Probabilités, XXVIII*, volume 1583 of *Lecture Notes in Math.*, pages 98–101. Springer, Berlin, 1994.

[Poq70] Jean-Baptiste (known as Molière) Poquelin. *Le bourgeois gentilhomme.* 1670. http://gallica.bnf.fr/scripts/ConsultationTout.exe?O=N023453.

[Pro03] Philip Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 2003. Second Edition.

[Sat99] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[Too97] Andre Toom. Tails in harnesses. *J. Statist. Phys.*, 88(1-2):347–364, 1997.

[Wil73] David Williams. Some basic theorems on harnesses. In *Stochastic analysis (a tribute to the memory of Rollo Davidson)*, pages 349–363. Wiley, London, 1973.

[Wil80] David Williams. Brownian motion as a harness. Unpublished, 1980.

[Zho92] Zhan Gong Zhou. Two-parameter harnesses and the generalized Brownian sheet. *Natur. Sci. J. Xiangtan Univ.*, 14(2):111–115, 1992.

[Zhu88] Xing Wu Zhuang. The generalized Brownian sheet and two-parameter harnesses. *Fujian Shifan Daxue Xuebao Ziran Kexue Ban*, 4(4):1–9, 1988.

[ZZ88] Run Chu Zhang and Xing Wu Zhuang. Two-parameter harnesses and a characterization of Brownian sheets. *Kexue Tongbao (Chinese)*, 33(22):1694–1697, 1988.